Math 3236 Statistical Theory

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$\hat{\theta}$ ML estimator

$$L(\theta, \underline{X}) = \prod_{i=1}^{N} f(X_i ; \theta)$$

X_i are a random sample

with distribution

$$f(x, \theta_0)$$

θ_0 is unknown.

$\hat{\theta}$

$$L(\hat{\theta}(\underline{X}), \underline{X}) = \sup_{\theta} L(\theta, X)$$

$$\hat{\theta}(\underline{X}) \xrightarrow{P} \theta_0$$

$$\hat{\theta}(\underline{X}) \simeq N\left(\theta_0, \frac{1}{\sqrt{n\,\mathbb{E}([\partial_\theta \ln f(X, \theta_0)]^2)}}\right)$$

$$I(\theta_0) = \mathbb{E}([\partial_\theta \ln f(X, \theta_0)]^2)$$

$I(\theta_0)$ Fisher information of

The r.v. X.

$$I(\theta_0) = \int_{-\infty}^{\infty} \left[\partial_\theta \ln f(x, \theta_0)\right]^2 f(x, \theta_0)\, dx$$

$$= -\int_{-\infty}^{\infty} \partial_\theta^2 \left[\ln f(x, \theta_0)\right] f(x, \theta_0)\, dx$$

$$= -\mathbb{E}\left(\partial_\theta^2 \ln f(X, \theta_0)\right)$$

$$\sqrt{n I(\theta_0)}\left(\hat{\theta} - \theta_0\right) \xrightarrow{d} N(0, 1)$$

$$I(\theta_0) = Var\left(\partial_\theta f(X, \theta_0)\right)$$

$$\underline{\theta} = (\theta_1, \theta_2)$$

$$I_{i,j}(\underline{\theta}) = Cov\left(\partial_i \ln f(X, \underline{\theta}), \partial_j \ln f(X, \underline{\theta})\right)$$

$(\hat{\theta}_1, \hat{\theta}_2)$ ML estimators

$$\sqrt{N}\left((\hat{\theta}_1, \hat{\theta}_2) - (\theta_{1,0}, \theta_{2,0})\right) \Rightarrow$$

$$\mathcal{N}\left(0, \frac{1}{I}\right)$$

Cramer - Rao Inequality

$T = r(\underline{X})$ is a statistics

X have distribution $f(x; \theta)$

$$Var_{\theta_0}(T) \geq \frac{[m'(\theta_0)]^2}{n\, I(\theta_0)}$$

$$m(\theta) = \int x\, f(x; \theta)\, dx = \mathbb{E}_\theta(X)$$

If $T = \hat{\theta}$ is unbiased estimator for θ $\Rightarrow$ $m(\theta) = \theta$

$$m'(\theta) = 1$$

$$var(\hat{\theta}) \geq \frac{1}{n\, I(\theta_0)}$$

$$\mathrm{Cov}\left(r(\underline{X})\, \partial_\theta \ln f_N(\underline{X}, \theta) \right) =$$

$$\int r(\underline{x})\, \partial_\theta \ln f_N(\underline{x}, \theta) \quad f_N(\underline{x}; \theta)\, dx =$$

$$f_N(\underline{x}, \theta) = \prod_{i=1}^{N} f(x_i; \theta)$$

$$\mathbb{E}\left(\partial_\theta \ln f_N(\underline{X}, \theta) \right) = 0$$

$$=$$

$$\int r(\underline{x})\, \partial_\theta f_N(\underline{x}, \theta)\, dx =$$

$$\partial_\theta \int r(\underline{x})\, \partial_\theta f_N(x, \theta)\, dx = \partial_\theta m(\theta)$$

$$\mathrm{Cov}\left(r(\underline{X}),\ \partial_\theta \ln f_N(\underline{X}, \theta) \right) = \partial_\theta m(\theta)$$

$$\mathrm{Cov}(X, Y)^2 \leq \mathrm{Var}(X)\, \mathrm{var}(Y)$$

$$\left[\partial_\theta m(\theta) \right]^2 \leq \mathrm{var}(r(\underline{X}))\, (N I(\theta))$$

Generalized Method of Moments

$X_i \qquad i = 1 \ldots N \qquad f(x; \theta_0)$

$m(x; \theta)$

$$\mathbb{E}_{\theta_0}(m(X, \theta)) = M(\theta_0, \theta)$$

$$\shortparallel$$

$$\int m(x,\theta) f(x; \theta_0)\, dx$$

$M(\theta, \theta) = 0 \qquad \forall \theta$

$M(\theta_0, \theta) \neq 0 \qquad \theta_0 = \theta$

$\mu(\theta) = \mathbb{E}_{\theta}(X)$

$$\mathbb{E}_{\theta_0}\left(X - \mu(\theta) \right) = M(\theta_0, \theta)$$

$m(x, \theta) = x - \mu(\theta)$

$M(\theta_0, \theta) = \theta_0 - \theta$

$$\frac{1}{N}\sum_{i=1}^{N} m(X_i;\theta) \approx \mathbb{E}_{\theta_0}(m(X,\theta))$$

$$\frac{1}{N}\sum_{i=1}^{N} m(X_i;\theta) = 0$$ a solution of this equation is close to a solution of

$$\mathbb{E}_{\theta_0}(m(X,\theta)) = 0$$

$$\Downarrow$$

$$M(\theta_0,\theta) = 0 \Longrightarrow \theta = \theta_0$$

<u>Simple example.</u>

$$\frac{1}{N}\sum_{i=1}^{N} m(X_i;\theta) = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu(\theta)) =$$

$$= \bar{X} - \mu(\theta)$$

$$\Downarrow$$

$$\hat{\theta} = \mu^{-1}(\bar{X}) \qquad \text{M.M.}$$

$$\mathbb{E}_{\theta_0}\left(\partial_\theta \ln f(X, \theta_0) \right) = 0$$

$$\mathbb{E}_{\theta_0}\left(\partial_\theta \ln f(X, \theta) \right) \neq 0 \qquad \theta \neq \theta_0$$

$$\Downarrow$$

$$\sum_{i=1}^{N} \partial_\theta \ln f(X_i, \theta) = 0$$

$$\partial_\theta \ln \prod_{i=1}^{n} f(X_i, \theta) = 0$$

$$\partial_\theta \ell(\theta, \underline{X}) = 0 \Longrightarrow \text{M.L.}$$

Are consistent and asymptotically normal.

$$\overline{X} = m_1(\theta)$$
$$\overline{X^2} = m_2(\theta) \qquad \overline{X^2} = \frac{1}{N} \sum_i X_i^2$$

$$\overline{X} - m_1(\theta) = 0$$
$$\overline{X^2} - m_2(\theta) = 0$$

θ That minimizes

$$\left(\bar{X} - m_1(\theta)\right)^2 + \left(\bar{X}^2 - m_2(\theta)\right)^2$$